

Matrix Multiplication

What You Need to Know

Matrix multiplication **AB** is only possible if the number of columns in matrix A ($m \times n$) equals the number of rows in matrix B ($n \times r$). In order to find the entries in AB, we multiply corresponding entries of m th row of A and r th column of B, as shown below in case of two square matrices.

If matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then $AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$.

Example 4: If matrix $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$ and matrix $B = \begin{pmatrix} 3 & -3 \\ 4 & 2 \end{pmatrix}$, find **AB** and **BA**.

$$AB = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2(3) + 3(4) & 2(-3) + 3(2) \\ 1(3) + (-1)(4) & 1(-3) + (-1)(2) \end{pmatrix} = \begin{pmatrix} 18 & 0 \\ -1 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3(2) + (-3)(1) & 3(3) + (-3)(-1) \\ 4(2) + 2(1) & 4(3) + 2(-1) \end{pmatrix} = \begin{pmatrix} 3 & 12 \\ 10 & 10 \end{pmatrix}$$

Note that $AB \neq BA$ which means that matrix multiplication is not commutative.

Example 5: If $A = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 2 & 1 \\ 5 & -3 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 0 & 5 \\ -4 & 7 \end{pmatrix}$, find the products

AB, BA, BC, CA, and AC.

$$AB = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \\ 5 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 8-15 & 4+9 & 2+0 \\ 4+25 & 2-15 & 1+0 \end{pmatrix} = \begin{pmatrix} -7 & 13 & 2 \\ 29 & -13 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & 2 & 1 \\ 5 & -3 & 0 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix} \text{ not possible (\# of columns in A } \neq \text{ \# of rows in B).}$$

$$BC = \begin{pmatrix} 4 & 2 & 1 \\ 5 & -3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 5 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} 8+0-4 & 4+10+7 \\ 10+0+0 & 5-15+0 \end{pmatrix} = \begin{pmatrix} 4 & 21 \\ 10 & -10 \end{pmatrix}$$

$$CA = \begin{pmatrix} 2 & 1 \\ 0 & 5 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 4+1 & -6+5 \\ 0+5 & 0+25 \\ -8+7 & 12+35 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 5 & 25 \\ -1 & 47 \end{pmatrix}$$

$$AC = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 5 \\ -4 & 7 \end{pmatrix} \text{ is not possible (\# of columns in A } \neq \text{ \# of rows in C).}$$

Determinants

What You Need to Know

Associated with square matrices is a real number called the determinant of a matrix and written as $\det(A)$ or $|A|$.

For the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = |A| = ad - bc$.

For the 3×3 matrix the expression for the determinant is more complex and it is expressed in terms of 2×2 determinants.

If matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$ then $\det(A) = |A| = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$.

The derivation of the expressions for 2×2 and 3×3 determinants is done by calculating an **inverse to the matrix**, which will be discussed in the next section. Matrices with determinant equal to 0 are called **singular**.

Example 6: If matrix $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -3 \\ 4 & 2 \end{pmatrix}$, find $\det(A)$ and $\det(B)$.

$$\det(A) = (2)(-1) - (1)(3) = -5$$

$$\det(B) = (3)(2) - (4)(-3) = 18$$

Example 7: If matrix $A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & -1 & 4 \\ 5 & 2 & 3 \end{pmatrix}$, find $\det(A)$.

$$\det(A) = 3 \begin{vmatrix} -1 & 4 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 4 \\ 5 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} = 3(-3-8) - 2(3-20) + 1(2+5) = 8$$

Example 8: If matrix $A = \begin{pmatrix} 4 & p \\ 5 & -3 \end{pmatrix}$ and matrix $B = \begin{pmatrix} 2 & -2 & 6 \\ 1 & q & 5 \\ -3 & 3 & -2 \end{pmatrix}$, find the

values of p and q for which each of the matrices is singular.

$$\det(A) = 0 \rightarrow -12 - 5p = 0 \rightarrow p = -\frac{12}{5}$$

$$\det(B) = 0 \rightarrow 2 \begin{vmatrix} q & 5 \\ 3 & -2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 5 \\ -3 & -2 \end{vmatrix} + 6 \begin{vmatrix} 1 & q \\ -3 & 3 \end{vmatrix} = 0$$

$$\text{Therefore } 2(-2q-15) + 2(-2+15) + 6(3+3q) = 0$$

$$-4q - 30 + 26 + 18 + 18q = 0$$

$$14q = -14$$

$$q = -1$$