Notes and jottings column

Choosing a Convergence Test

You will not always be told which convergence test to use; the following notes give you some advice on choosing the most suitable.

As a rough guide, the ratio test should be tried whenever u_n contains:

> The ratio test won't *necessarily* work in all these cases, but it is worth trying.

- Factors with n as a power (this includes *all* the power series)
- Factors containing factorials of functions of n

Examples of these are:

$$u_n = nx^{n+1};\ u_n = \frac{2^n + n}{5^n + 4n^3};\ u_n = \frac{n^2}{(2n+1)!}$$

The ratio test will *never* work when u_n is a rational function of n because the limit will always be 1. However, the comparison test will always be successful.
Examples of these are:

$$u_n = \frac{3n^2 + n}{n^4 + 1};\ u_n = \frac{n+1}{3n^2 + 2n}$$

The comparison test is also a good test when u_n is a quotient containing *roots* of polynomials.

Clearly the integral test can only be used if f(n) can be integrated – and in many examples it cannot. However, if the integration can be done, the integral test is worth at try.

Test whether the following series converge:

i) $\sum \frac{2^{n+7}}{3^n - 1}$

ii) $\sum \frac{2^n (n!)^2}{(2n)!}$

i) This looks like a ratio test. Unfortunately, the ratio does not simplify further than $\frac{2 \times (3^n - 1)}{3^{n+1} - 1}$, and we cannot find a limit. However, a comparison with $u_n = \frac{2^n}{3^n}$ looks a good bet. Using the limit comparison test we get:

$$\frac{2^{n+7}}{3^n - 1} \times \frac{3^n}{2^n} = 2^7 \times \frac{3^n}{3^n - 1} \to 2^7 \text{ as } n \to \infty$$

Since $\sum \frac{2^n}{3^n} = \sum \left(\frac{2}{3}\right)^n$ it is a geometric series. Hence it converges, and so too does $\sum \frac{2^{n+7}}{3^n - 1}$.

ii) This *definitely* looks like a ratio test!

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}((n+1)!)^2}{(2n+2)!} \times \frac{(2n)!}{2^n (n!)^2} = \frac{2(n+1)^2}{(2n+2)(2n+1)} = \frac{(n+1)}{(2n+1)}$$

This ratio tends to ½ as $n \to \infty$, so the series converges.
