Distances in Three Dimensions

Notes and jottings column

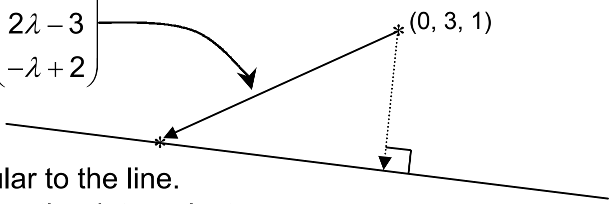
Distance between two points: We find the vector between the two points (using $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$) and then use Pythagoras' theorem in three dimensions to find the length of the vector. For example, the vector from (1, -2, 4) to (4, 2, 2) is $\begin{pmatrix}4\\2\\2\end{pmatrix} - \begin{pmatrix}1\\-2\\4\end{pmatrix} = \begin{pmatrix}3\\4\\-2\end{pmatrix}$.

The length of this vector is $\sqrt{(3^2 + 4^2 + (-2)^2)} = \sqrt{29}$. This is therefore the distance between the points.

Distance from a point to a line: This is the most involved of the various methods, and must be learnt. Suppose we want to find the distance from the point (0, 3, 1) to the line $\boldsymbol{r} = \begin{pmatrix}-2\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\-1\end{pmatrix}$.

Now, any point on the line has coordinates $(-2 + 3\lambda, 2\lambda, 3 - \lambda)$. So the vector from (0, 3, 1) to a general point on the line is:

$$\begin{pmatrix}-2+3\lambda\\2\lambda\\3-\lambda\end{pmatrix} - \begin{pmatrix}0\\3\\1\end{pmatrix} = \begin{pmatrix}3\lambda-2\\2\lambda-3\\-\lambda+2\end{pmatrix}$$

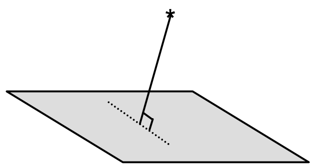


To find the shortest distance, we want this vector to be perpendicular to the line.
We can find the value of λ by using the dot product.

$$\begin{pmatrix}3\lambda-2\\2\lambda-3\\-\lambda+2\end{pmatrix} . \begin{pmatrix}3\\2\\-1\end{pmatrix} = 0 \Rightarrow (9\lambda-6)+(4\lambda-6)+(\lambda-2)=0 \Rightarrow \lambda = 1$$

Thus the perpendicular vector to the line is $\begin{pmatrix}3-2\\2-3\\-1+2\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ which has length $\sqrt{3}$, and this is the distance from the point to the line.

Distance from the origin to a plane: This little trick is used in each of the next four calculations, so it's a good one to know. For the plane $ax + by + cz = d$, simply divide d by the length of the normal vector, which is $\sqrt{a^2+b^2+c^2}$. For example, the distance from the origin to the plane $2x - 3y + z = 5$ is $\dfrac{5}{\sqrt{14}}$.



Distance between parallel planes: All we need to do to find their distance from each other is find the distance of each from the origin, and find the difference.

It's easy to see if two planes are parallel – they have the same normal vector, and hence the same LHS of the equation (or multiples of each other).

---

eg: Find the distance from $x + 2y + z = 7$ to $x + 2y + z = 5$.

The distance from the first plane to the origin is $\dfrac{7}{\sqrt{6}}$

The distance of the second plane from the origin is $\dfrac{5}{\sqrt{6}}$

Thus the distance between them is $\dfrac{7}{\sqrt{6}} - \dfrac{5}{\sqrt{6}} = \dfrac{2}{\sqrt{6}} = 0.816$

If one of the distances turns out negative, the plane is on the other side of the origin; the distances will be added.