Notes and jottings column

Applications: A differential is a rate of increase or decrease, and this is always the starting point for practical applications. Suppose, for example, that a town's rate of population increase is proportional to the population; this leads to the equation $\frac{dP}{dt} = kP$. To solve this, we divide by *P*, and continue as follows: $\int \frac{1}{P} dP = \int k dt \Rightarrow \ln P = kt + c$. In this sort of situation it is normal to make *P* the subject: $P = e^{kt + c}$ or $P = Ae^{kt}$. With two constants two find, we need two pieces of information. Suppose we are given that that $t = 0$ represents the year 1990 and the population was then 3000. We are also told that in 2000, the population was 3500.

Then: $2000 = Ae^0 \Rightarrow A = 2000$

$3500 = 2000 \times e^{10k} \Rightarrow 1.75 = e^{10k} \Rightarrow k = 0.0560$

Thus, the full solution is $P = 2000e^{0.0560t}$.

> We can use the solution to solve further questions such as "what will the population be in 2005?" or "when will the population reach 10000?"
> *Answers: 4632 and 2019.*

Speed and acceleration are other typical quantities that form the basis of differential equations.

> Velocity = $\frac{ds}{dt}$
> Acceleration = $\frac{dv}{dt}$

The velocity of an object, *v*, at time *t* is given by $v = 3e^{-t/2}$. Find how far it travels from *t* = 0 until *t* = 2.

$\frac{ds}{dt} = 3e^{-t/2} \Rightarrow \int ds = \int 3e^{-t/2} dt$ so $s = -6e^{-t/2} + c$

When $t = 0$, $s = -6 + c$. When $t = 2$, $s = -2.21 + c$. Thus the distance travelled is $(-2.21 + c) - (-6 + c) =$ **3.79**

Often, questions are more algebraic.

The velocity of an object, *v*, at time *t* is given by $v = ke^{2t}$. Find the distance travelled between *t* = 0 and *t* = *T*.

Using similar working to the above example, we find that $s = 0.5ke^{2t} + c$.

When $t = 0$, $s = 0.5k + c$

When $t = T$, $s = 0.5ke^{2T} + c$

So, distance travelled = $(0.5ke^{2T} + c) - (0.5k + c) =$ **$0.5k(e^{2T} - 1)$**

A sample of radioactive material decays at a rate which is proportional to its mass, *m*. Given that 50g of material decays to 48g in 10 years, find the "half-life", ie the time taken for a given mass to halve.

You will need a minus sign on the right hand side of the differential equation because the rate of change is negative. You are given two facts (although it only looks like one), so you can find both the constant of proportionality and the constant of integration.

$m = 50e^{-0.00408t}$, 170 years

YOU SOLVE

Homogeneous equations: Any equation which reduces to the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ is called *homogeneous* and can be solved by the substitution $y = vx$. This *always* gives a LHS of $v + x\frac{dv}{dx}$. eg:

$\frac{dy}{dx} = \frac{x - y}{x} \Rightarrow v + x\frac{dv}{dx} = \frac{x - vx}{x} \Rightarrow x\frac{dv}{dx} = 1 - 2v$. The variables are now separable, and the solution is $-\tfrac{1}{2}\ln(1 - 2v) = \ln x + c$. Further algebraic manipulation and then resubstitution of $v = y/x$ gives the final solution: $x^2 - 2xy = A$.

> How do you know when to use this method? If the variables do not separate directly, then it must be a homogeneous equation.