Simplify: $8 = 3 \times 152 - 8 \times 56$
Substitute: $8 = 3 \times 152 - 8 \times (208 - 1 \times 152)$
Simplify: $8 = -8 \times 208 + 11 \times 152$
Substitute: $8 = -8 \times 208 + 11 \times (568 - 2 \times 208)$
Simplify: $8 = 11 \times 568 - 30 \times 208$
Thus, $x = 11$ and $y = 30$.

This procedure is called Euclid's extended algorithm

The questions which arise are:

- Are there more solutions?
- How do we find solutions when the RHS is a multiple of *d*?

The general solution of $ax + by = c$ is based on the solution of $ax + by = d$ found by using the method above. Let us call the solution values x_d and y_d, ie $x_d = 11$ and $y_d = 30$ in this case. The general solution of $ax + by = c$ is given by the formulae:

$$x = \frac{c}{d}x_d - t\frac{b}{d}, \quad y = \frac{c}{d}y_d + t\frac{a}{d}, \text{ where } t \in \mathbb{Z}$$

In other words, different values of *t* will give different solutions.

Find the general solution of $6x - 15y = 27$
First apply Euclid's algorithm to 6 and –15:

$-15 = -3 \times 6 + 3$
$6 = 2 \times 3$

So gcd(6, -15) = 3 and, since 3|27, there will be solutions.
There is only one step to reverse, and this gives $3 \times 6 - 1 \times 15 = 3$. So we have $a = 6$, $b = -15$, $d = 3$, $c = 27$, $x_d = 3$ and $y_d = 1$. The general solution will be:

$$x = \frac{27}{3} \times 3 - t\frac{(-15)}{3} = 27 + 5t, \quad y = \frac{27}{3} \times 1 + t\frac{6}{3} = 9 + 2t$$

Note that you could divide through by 3 first, but with large values of *a*,*b* and *c*, this may not be obvious.

Be very careful to put minus signs in all the right places.

If $t = 0$, we have $x = 27$ and $y = 9$
If $t = 1$, we have $x = 32$ and $y = 11$
If $t = -1$, we have $x = 22$ and $y = 7$
You might like to check that these solutions work. Solutions may be subject to constraints – for example, if all the solutions had to be positive, work out what constraints that would place on *t*.

$t > -4$

Practice questions:

1. If $a|b$ and $b|c$, $a, b, c \in \mathbb{Z}$, prove that $a|c$.
2. Write the following numbers as the product of their prime factors: 30, 32, 38, 111, 120.
3. Use prime factorisations to find gcd(92, 120) and lcm(18, 60).
4. Use Euclid's algorithm to calculate gcd(352, 1540) .
5. Prove that $8n + 3$ and $5n + 2$, $n \in \mathbb{N}$, are relatively prime.
6. Prove that gcd(a, $a + b$) = gcd(a, b)
7. Find the general solution of the equation $3x + 4y = 1$ given that $x,y \in \mathbb{Z}$ and $|x|,|y| < 50$.

Answers:

1. $b = pa$, $c = qb$ ($p,q \in \mathbb{Z}$). So $c = qb = qpa$, thus $a|c$.
2. $2 \times 3 \times 5$, 2^5, 2×19, 3×37, $2^3 \times 3 \times 5$
3. 4, 180
4. 44
5. $8n + 3 = 1 \times (5n + 2) + (3n + 1)$, $5n + 2 = 1 \times (3n + 1) + (2n + 1)$, $3n + 1 = 1 \times (2n + 1) + n$, $2n + 1 = 2 \times (n) + 1$
   Thus gcd = 1 and $8n + 3$ and $5n + 2$ are relatively prime
6. If $d = \gcd(a,b)$, then for some $p,q \in \mathbb{Z}$ $a = pd$ and $b = qd$. Thus $a + b = pd + qd = d(p + q)$. So $d|a$ and $d|(a + b)$. p and q are relatively prime, so p and $p + q$ are as well. So there is no higher common divisor than d. Thus gcd(a, $a + b$) = gcd(a, b)
7. $x = -1 - 4t$, $y = 1 + 3t$, $t \in \mathbb{Z}$ such that $t < 12$