There are some further formulae for planar graphs which you are required to know.

> For a connected planar simple graph with n (>2) vertices and e edges, $e \leq 3n - 6$.

> For a connected planar simple graph with n vertices and e edges, $e \leq 2n - 4$.

> A connected planar simple graph will have at least one vertex of degree 5 or less.

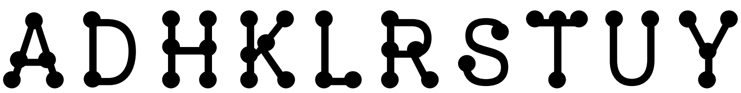
Note that the graphs must be *simple* (no loops or multiple edges).

The proofs of the first two theorems are on page 15. For the third, consider that every vertex in the graph *does* have degree 6 or more. It follows that $2e \geq 6n$, hence $e \geq 3n$. But $e \leq 3n - 6$, leading to a contradiction.

The converses of these theorems are not true: for example, a graph with a degree 3 vertex is not necessarily planar. However, if it does *not* have at least one vertex of degree 5 or less, it cannot be planar. Try using the first theorem to show that K_5 is not planar, and the second to show $K_{3,3}$ is not planar. What happens if you try these proofs the other way around?

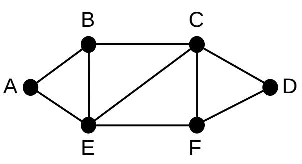
Isomorphism: The famous schematic map of the London Underground railways is not a map in the sense that it shows distances and directions. But in terms of the relationships between the stations (for example, their order on a particular line), and where lines intersect, the schematic map is exactly the same as the true map. The two maps are *isomorphic* to each other. Basically the connectivity of two isomorphic graphs is the same. Clearly, they must have the same number of vertices of each order: in addition, they must be connected together in the same manner.

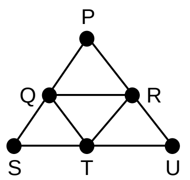
Below are 10 letters of the alphabet, drawn as connected graphs. Which ones are isomorphic to each other?



Answers:
A and R. U and S. Y and T. H and K.

Why are the two graphs below not isomorphic? Show how moving one edge on the second graph can make them isomorphic.





Each has 6 vertices, 9 edges and 5 faces. However, the first graph has vertices of order 2, 2, 3, 3, 4, 4 whereas the second graph has vertices of order 2, 2, 2, 4, 4, 4. So the connectivity is different and the graphs are not isomorphic.

One of the vertices of order 4 must become order 3 as must one of the vertices of order 2. I began by imagining the second graph rotated anti-clockwise through 90° so that PQR becomes AEB. A bit of trial and error, and the isomorphism is achieved by disconnecting edge UR and reconnecting as US. The points then correspond as follows: $A \leftrightarrow P$, $B \leftrightarrow R$, $C \leftrightarrow T$, $D \leftrightarrow U$, $E \leftrightarrow Q$, $F \leftrightarrow S$.
