Isomorphism and infinite groups: Proofs with infinite groups will involve general elements. For example:

By considering the bijection $\Phi:\mathbb{R} \to \mathbb{R}^+$ where $\Phi(x) = e^x$, show that the group of real numbers under addition is isomorphic to the group of positive real numbers under multiplication.

What are we being asked to show? The diagram illustrates in general terms what we must do.

$$\begin{tikzcd}
 & a,b \arrow[dl] \arrow[dr] & \\
f(a),\ f(b) \arrow[dr] & & a * b \arrow[dl] \\
 & f(a) \sim f(b) = f(a * b) &
\end{tikzcd}$$

Take two elements of $\mathbb{R}$, a and b. The two corresponding elements in $\mathbb{R}^+$ are e^a and e^b. Adding the original elements gives $a + b$. What we need to show is that there is a one-to-one correspondence between e^{a+b} and $e^a \times e^b$. Clearly, by the rules of indices, $e^{a+b} \equiv e^a \times e^b$. Hence the isomorphism is proved.

Sometimes a question involves the isomorphism of a group onto itself, in which case we will only be dealing with one operation. That is, we need to show that $\Phi(x*y) = \Phi(x) * \Phi(y)$. This principle is used in the following question:

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a) In any group, show that if the elements x, y, and xy have order 2, then $xy = yx$.

This is the sort of question where we must just write things down and see what happens. If each of those elements has order 2, then:

$x^2 = e$ so $xx = e$

$y^2 = e$ so $yy = e$

$(xy)^2 = e$ so $xyxy = e$

Note that it is often worthwhile expanding the "squares" like this. Also note that $(xy)^2 \neq x^2y^2$; this is not ordinary algebra!

Now the only way we can use the xx and the yy in the third expression is to put an x in front and a y at the end. This is what we get:

$x(xyxy)y = xey$

$(xx)yx(yy) = xey$ (Using the associative property)

$eyxe = xey$

So $yx = xy$ qed

b) Show that the inverse of each element in a group is unique.

This is a standard proof which you are required to know. There are several different ways to set about it. In each case, we start by assuming that there are two inverses. For example:

If an element a has inverses b and c, then:

$ab = ac = e$

By the left cancellation law, if $ab = ac$ then $b = c$

But this is not enough. The definition of an inverse is that $aa^{-1} = a^{-1}a = e$

$ba = ca = e$

By the right cancellation law, if $ba = ca$ then $b = c$.

Thus no element can have more than one inverse.

c) Let G be a group. Show that the correspondence $x \leftrightarrow x^{-1}$ is an isomorphism from G onto G if and only if G is Abelian.

The "if and only if " means we must work the proof both ways. Remember that "Abelian" means the operation is commutative.

To be an isomorphism, $(x * y)^{-1} = x^{-1} * y^{-1}$.

But $(x * y)^{-1} = y^{-1} * x^{-1}$, so we require $x^{-1} * y^{-1} = y^{-1} * x^{-1}$, hence G must be Abelian.

$(x * y)^{-1} = y^{-1} * x^{-1}$ is a standard rule for inverses.

If G is Abelian then $x * y = y * x$, hence:

$(x * y)^{-1} = (y * x)^{-1} = x^{-1} * y^{-1}$, hence G is isomorphic.