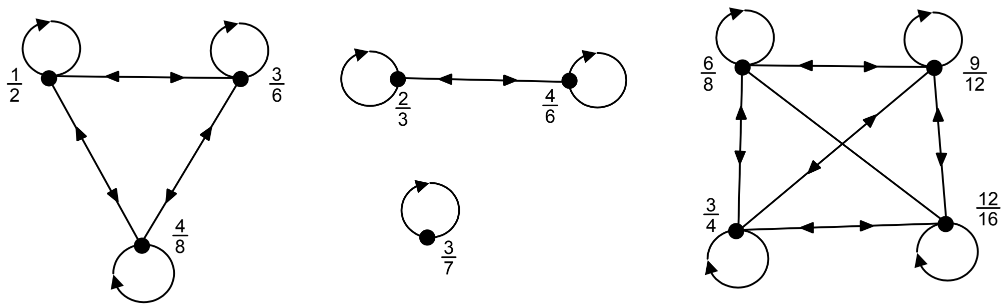

Note the following points:

- Each element has a loop to itself. R is reflexive.
- Each pair of elements has a two-way arrow. R is symmetric.
- Whenever a path follows two sides of a triangle, the single path from start to end is also present. R is transitive.
- These properties also apply when there are only one or two fractions present in a group. For example the arrows from $\frac{2}{3}$ to $\frac{4}{6}$ to $\frac{4}{6}$ also represent the transitive property of R.

When a relation is reflexive, symmetric and transitive, it is called an *equivalence relation.* Related elements are said to be *equivalent* and the various groupings are called *equivalence classes*. Thus the equivalence classes always contain equivalent elements. The division into disjoint equivalence classes is called a *partition* of the set.

Let S = {(x, y) | x, $y \in \mathbb{R}$}, and let (a, b), (c, d) $\in$ S. Define the relation # on S as follows:

$$(a, b) \# (c, d) \Leftrightarrow a^2 + b^2 = c^2 + d^2$$

a) Show that # is an equivalence relation.

$a^2 + b^2 = a^2 + b^2 \Rightarrow$ # is reflexive

If $a^2 + b^2 = c^2 + d^2$ then $c^2 + d^2 = a^2 + b^2 \Rightarrow$ # is symmetric.

If $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$ then $a^2 + b^2 = e^2 + f^2 \Rightarrow$ # is transitive.

Hence # is an equivalence relation.

b) Find all the ordered pairs (x, y) where (x, y) # (1, 2)

This translates to the equation $x^2 + y^2 = 5$. Clearly there are an infinite number of ordered pairs for which this is true so we must express the solution algebraically rather than providing a list.

The set of ordered pairs is $\{(x, y) \mid y = \pm\sqrt{(5 - x^2)}), |x| < \sqrt{5}, x \in \mathbb{R})$

c) Describe the partition created by this relation on the (x, y) plane.

When asked to describe a partition, we need to define the equivalence classes created by the relation. In this case, we are asked to consider the ordered pairs as points on the plane. The answer to (b) gives us the clue – where are all the points (x, y) such that $x^2 + y^2 = 5$? On the circle, centre the origin, radius $\sqrt{5}$. These points will form one equivalence class. Thus we deduce that the equivalence classes created will be points on all possible circles, centre the origin.

Using a matrix: Suppose A = {*a, b, c, d, e, f*} and the relation R on A is defined by the matrix shown right. If the rows and columns are headed *a* to *f* then the second 1 in the top row, say, shows that *a*R*c*. It is given that R is transitive. R is reflexive because there are 1's down the leading diagonal; it is symmetric because the matrix is symmetrical about the leading diagonal. So R is an equivalence relation, the equivalence classes being {*a, c, e*}, {*b, d*} and {*f*}. Note that we could easily draw an arrow diagram directly from the matrix.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$