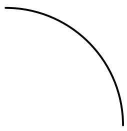
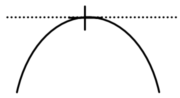
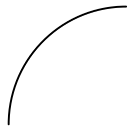
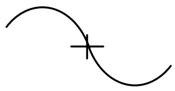
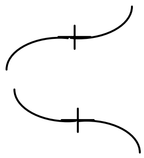
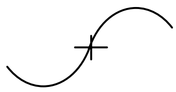
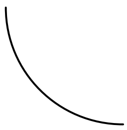
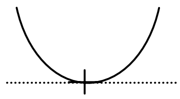
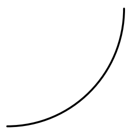
Notes and jottings column

Second Derivative

Notation: When a function is differentiated a second time, use the notation $\frac{d^2y}{dx^2}$ or *f''(x)*.

Interpretation: The first derivative gives us the gradient function, so the second derivative gives us the "rate of change of gradient" function. If, for example, $f''(3) = 2$ this means that when $x = 3$, the gradient of the graph is increasing at a rate of 2 (for every increase in x of 1). It does not necessarily mean that the gradient itself is positive – only that it is increasing. This tells us about the shape of the curve. The diagram below shows what happens for various values of the first and second derivatives and covers every possible point on any curve.

	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$
$\frac{d^2y}{dx^2} < 0$		 MAXIMUM	
$\frac{d^2y}{dx^2} = 0$			
	POINTS OF INFLEXION		
$\frac{d^2y}{dx^2} > 0$		 MINIMUM	

Note the following:

- Points of inflexion occur where $f''(x) = 0$, but the gradient at a point of inflexion is not necessarily 0.
- The sign of the second derivative at a turning point identifies the nature of the point: a maximum if $f''(x) < 0$, a minimum if $f''(x) > 0$.

Find and identify the turning point on the graph of $f(x) = xe^{-x}$

We need to use the product rule to differentiate.

$$f'(x) = x(-e^{-x}) + 1\times(e^{-x}) = -xe^{-x} + e^{-x}$$

For turning points, $f'(x) = 0$, so $-xe^{-x} + e^{-x} = 0 \Rightarrow e^{-x}(-x + 1) = 0 \Rightarrow$ **$x = 1$**

We also need the y-coordinate. When $x = 1$, $f(1) = e^{-1}$, so TP is at **$(1, e^{-1})$**

The y-coordinate has been given as e^{-1}. We could also have used $1/e$ or worked it out as the decimal 0.368….

To identify the turning point we must differentiate again. We note that the first part of the function *f'(x)* is the same as *f(x)*, but with a minus sign. So it will give the same derivative, with a minus sign.

$$f''(x) = -(-xe^{-x} + e^{-x}) - e^{-x} = xe^{-x} - 2e^{-x}$$

When $x = 1$, $f''(x) = 1 \times e^{1} - 2e^{1} = -e$. So $f''(x) < 0 \Rightarrow$ maximum

The turning point is at $(1, e^{-1})$ and it is a maximum

(Answers to this type of question can be easily checked on a calculator)